preconditioner evaluation. An immediate benefit of this technique is a marked reduction in the preconditioner memory requirements. This is reflected in the reduced number of nonzero diagonals (or block diagonals) required for the same level of fill in. In all cases, the average GMRES iterations diminished or stayed the same, indicating that the overall preconditioner quality was not negatively impacted, whereas CPU efficiency was improved.

Geometry Effects: Problem 2

Performance data for the idealized combustion chamber model problem using a 120×70 uniform grid are presented in Table 3 for $Re = 60$ and $Ma = 0.14$. These calculations were initialized from an interpolated 60×35 grid solution. The pseudotime step control parameters were again set to the values used earlier so that $CFL^0 \approx 1.4$ and $CFL^{max} \approx 2800$. The ILU subdomain solvers were not effective for any of the partition selections. As with the first model problem, stripwise cuts across the flow channel were more effective than cuts along the flow channel. As shown in Table 3, both the low-order and high-order discretizations were used in the preconditioner evaluation for comparison purposes. Using the first-order upwind discretization scheme in the preconditioner evaluation was more effective than using the flux limited QUICK scheme in terms of both computer memory requirements and CPU times. This was the case despite that, at $Re = 60$, the flow reattachment length of the flux limited QUICK solution is approximately 53% longer than that obtained using first-order upwinding.

Summary

This Note presents pseudotransient Newton–Krylov–Schwarz solutions to two idealized low-Mach-number combustion problems. Schwarz methods offer effective and flexible preconditioning options for problems of this type, while also providing an avenue toward parallelism. BILU subdomain solvers generally provided a good compromise between subdomain solver quality and memory requirements. Preconditioning with a low-order discretization proved to be a wise choice in terms of both computer memory requirements and algorithm performance.

Acknowledgments

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Improved Optimality Criterion Algorithm for Optimal Structural Design

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I. Introduction

THE optimal design of structures with frequency constraints is extremely useful in manipulating dynamic characteristics in a variety of ways.[1] References 2 and 3 first pointed out that the singularity of eigenvalue derivatives with respect to the design vector does not allow use of the Kuhn–Tucker condition with multiple frequency constraints. Recently, Czyz and Lukasiewicz[4] put forward the optimality criteria using Lagrange multipliers. The strategy of these algorithms in overcoming the singularity is that the variation (or increment) $\delta \boldsymbol{h}$ of design vector $\boldsymbol{h}$ is confined to a subspace $\Delta \boldsymbol{H}$ in which the multiplicity of the eigenvalue does not change ($\delta \boldsymbol{h} \in \Delta \boldsymbol{H}$). Reference 5 takes damping into account, but once the repeated eigenvalues appear this method will be doomed to failure.

Various optimal structural methods under multiple eigenvalue constraints require knowledge of corresponding modes and sensitivities of the (progressively improved) structure. Hence, after almost every iteration improvement, modes and sensitivity would be reanalyzed. To save computational cost and hasten the optimal design, Refs. 6 and 7 posed the substructural sensitivity synthesis techniques; unfortunately, their formulas are confined to simple eigenvalues.

The purpose of this Note is to develop a highly efficient algorithm for structural optimization under multiple frequency constraints, making use of the generalized variations $\delta \boldsymbol{q}$ of design vectors $\boldsymbol{h}$ to automatically achieve the optimality search of $\delta \boldsymbol{h} \in \Delta \boldsymbol{H}$. This not only ensures that the Kuhn–Tucker condition holds but also excludes additional Lagrange multipliers. We use the improved substructural synthesis suitable for repeated eigenvalue sensitivity analysis of general nondefective vibratory systems[8,9] to quickly draw the required simple or repeated eigenvalue modes and their sensitivities of a complex structure after every iteration.

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II. Substructural Sensitivity Synthesis

A. Modal Synthesis

The entire structure is divided into two kinds of substructures: modifiable and unmodifiable, denoted by superscripts β and γ, respectively. Craig's condensation transformation

$$y = Tx \tag{1}$$

makes the entire structure finite element node displacements $y^t = [\mathrm{row}(^{(\alpha)}u_i^t)v^t]$ be expressed by modal coordinates $x^t = [\mathrm{row}(^{(\alpha)}\xi_k^t)\ v^t]$, where the transform matrix is

$$T = \begin{bmatrix} \mathrm{diag}\left(^{(\alpha)}\Phi_{ik}\right) & \mathrm{col}\left(^{(\alpha)}C_{ij}B_{jv}\right) \\ 0 & I_{vv} \end{bmatrix}, \qquad \alpha = \beta, \gamma \tag{2}$$

The substructural constraint modes, retained fixed-interface eigenpairs (Φ_{ik}, Ω_k), and condensation mass and stiffness matrix formulas are given as follows:

$$C_{ij} = -k_{ii}^{-1}k_{ij} \tag{3}$$

$$k_{ii}\Phi_{ik} = m_{ii}\Phi_{ik}\Omega_k, \qquad \Phi_{ik}^t m_{ii}\Phi_{ik} = I_k \tag{4}$$

with

$$\mu = \begin{bmatrix} \mathrm{diag}\left(^{(\alpha)}I_k\right) & \mathrm{col}\left(^{(\alpha)}a_{kj}B_{jv}\right) \\ \mathrm{sym} & \Sigma_\alpha^* \bar{m}_{jj} \end{bmatrix} \tag{5a}$$

$$\kappa = \begin{bmatrix} \mathrm{diag}\left(^{(\alpha)}\Omega_k\right) & 0 \\ \mathrm{sym} & \Sigma_\alpha^* \bar{k}_{jj} \end{bmatrix} \tag{5b}$$

where Σ^* is assembly summation

$$\Sigma^* = \Sigma B_{jv}^t()B_{jv}, \qquad \Sigma_\alpha^* = \Sigma_\beta^* + \Sigma_\gamma^*$$

and

$$a_{kj} = \Phi_{ik}^t(m_{ii}C_{ij} + m_{ij}) \tag{6}$$

Thus, we derive the reduced eigenvalue problem in terms of coordinates

$$\kappa X = \mu X \Lambda, \qquad X^t \mu X = I \tag{7}$$

The structural physical modes are determined by the transform of Eq. (1):

$$Y = TX \tag{8}$$

B. Eigenvalue Directional Derivative

Generally, an eigenproblem is expressed by

$$K(h)z = \theta M(h)z, \qquad z^t M(h)z = 1 \tag{9}$$

where $h \in R^n$ is the design vector and z is the eigenvector corresponding to eigenvalue θ.

In Ref. 8, it was proven that, if eigenvalues $\theta(h)$ (at $h = h_0$) have a repeated number $r > 1$, the eigenvalues are only directionally differentiable and, along δh, their directional derivatives are eigenvalues $\delta\Theta = \mathrm{diag}(\delta\theta_1, \delta\theta_2, \ldots, \delta\theta_r)$ of the following r-order standard eigenproblem:

$$AR = R\delta\Theta, \qquad R^t R = I_r \tag{10}$$

where $\delta\theta_s$ $(s = 1, 2, \ldots, r)$ represents the sth increment of r repeated eigenvalue $\theta(h_0)$ caused by δh.

In Eq. (10)

$$A = Z_0^t \Delta Z_0, \qquad \Delta \stackrel{\mathrm{def}}{=} [\delta K - \theta(h_0)\delta M]|_{h_0} \tag{11}$$

where Z_0 is any r orthonormalized eigenvalue set of Eq. (9) with respect to $\theta(h_0)$. Multiplied by the eigenvector R of Eq. (10),

$$Z = Z_0 R \tag{12}$$

is another orthonormalized eigenvector set of Eq. (9) with respect to its eigenvalue $\theta(h_0)$, but it is Gateaux differentiable and also has

$$Z^t \Delta Z = \delta\Theta \tag{13}$$

The directional derivative of Z has a more complex form[8]:

$$\delta Z = \delta\tilde{Z} + Zd \tag{14}$$

C. Sensitivity Synthesis

Calculating variations of the transform matrix and condensation mass and stiffness matrices expressed by Eqs. (2) and (5) along δh gives

$$\delta T = \begin{bmatrix} \mathrm{diag}\left(^{(\beta)}\delta\Phi_{ik}\right) & 0 & \mathrm{col}\left(^{(\beta)}\delta C_{ij}\cdot B_{jv}\right) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

and

$$\delta\mu = \begin{bmatrix} 0 & 0 & \mathrm{col}\left(^{(\beta)}\delta a_{kj}\cdot B_{jv}\right) \\ & 0 & 0 \\ \mathrm{sym} & & \Sigma_\beta^*\delta\bar{m}_{jj} \end{bmatrix}$$

$$\delta\kappa = \begin{bmatrix} \mathrm{diag}\left(^{(\beta)}\delta\Omega_k\right) & 0 & 0 \\ & 0 & 0 \\ \mathrm{sym} & & \Sigma_\beta^*\delta\bar{k}_{jj} \end{bmatrix} \tag{16}$$

where $\delta C_{ij}, \delta a_{kj}, \delta\bar{m}_{jj}$, and $\delta\bar{k}_{jj}$ can be derived by the general derivation rules of the function.

Note that (Φ_{ik}, Ω_k) are eigenpairs of constraint substructure Eq. (4) only if in Sec. II.B, K, M, Θ, and Z in Eqs. (9) and (10) are changed by k_{ii}, m_{ii}, Ω_k, and Φ_{ik}. We immediately can compute eigenpairs variations $(\delta\Phi_{ik}, \delta\Omega_k)$ along δh according to the relevant equations in Sec. II.B. Similarly, using known $\delta\mu$ and $\delta\kappa$, we can compute variations δX and $\delta\Lambda$ and then with the aid of Eqs. (2) and (15) obtain the eigenvector sensitivities of the entire structure:

$$\delta Y = T\cdot\delta X + \delta T\cdot X \tag{17}$$

III. Generalized Optimality Criterion Method

The optimality problem is stated as follows.[5]

Minimize

$$W(h) = \sum_{k=1}^{n} h_k w_k \tag{18}$$

under multiple frequency constraints

$$\bar{\lambda}_i^l \le \lambda_i \le \bar{\lambda}_i^u, \qquad i = 1, 2, \ldots, m \tag{19}$$

Thus, the Lagrange function is formed as follows:

$$L(h, \nu) = W(h) + \sum_{i=1}^{m}(\nu_{if} f_i - \nu_{ig} g_i) \tag{20}$$

where ν_{if} and ν_{ig} are Lagrange multipliers.

When $r_i > 1$, for λ_i, is not Fréchet differentiable, the Kuhn–Tucker condition cannot be used. For its validity, Czyz and Lukasiewicz[4] gave a set of (additional) constraint equations of δh:

$$\sum_{k=1}^{n} a_{ij}^k \delta h_k = 0, \qquad j > i = 1, 2, \ldots, r-1$$

$$\sum_{k=1}^{n} \left(a_{ii}^k - a_{11}^k\right)\delta h_k = 0, \qquad i = 2, \ldots, r \tag{21a}$$

where

$$a_{ij}^k = z_i^t\left[\frac{\partial K}{\partial h_k} - \theta(h_0)\frac{\partial M}{\partial h_k}\right]z_j, \qquad k = 1, 2, \ldots, n$$

It is the necessary condition that ensures $\delta h \in \Delta H$.

The equivalent matrix expression of Eq. (21a) is

$$D\delta h = 0 \tag{21b}$$

where D is a $[(r-1)(r+2)/2]n$ matrix. If the nullification of D is p, then Eq. (21b) has a p linear independent nontrivial solution $\delta h^1, \delta h^2, \ldots, \delta h^p$, which spans the null space of D.

An arbitrary solution of Eq. (21) can be rewritten as

$$\delta h = C \cdot \delta q \tag{22}$$

Substituting Eq. (22) into the variation of the Lagrange function gives

$$\delta L(h, \nu) = \frac{\partial W}{\partial h}\delta h + \sum_{i=1}^{m}(\nu_{if} - \nu_{ig})\frac{\partial \lambda_i}{\partial h}\delta h$$

$$= \frac{\partial W}{\partial h}C \cdot \delta q + \sum_{i=1}^{m}(\nu_{if} - \nu_{ig})\frac{\partial \lambda_i}{\partial h}C \cdot \delta q \tag{23}$$

So far, the Kuhn–Tucker condition can be used:

$$\frac{\partial W}{\partial h}C + \sum_{i=1}^{m}(\nu_{if} - \nu_{ig})\frac{\partial \lambda_i}{\partial h}C = 0 \tag{24}$$

with

$$\nu_{if}, \nu_{ig} \geq 0, \qquad \nu_{if} f_i = \nu_{ig} g_i = 0, \qquad i = 1, 2, \ldots, m \tag{25}$$

From Eq. (24), one obtains

$$v_k + s_k = 0, \qquad k = 1, 2, \ldots, p \tag{26a}$$

or

$$-s_k/v_k = 1 \tag{26b}$$

An iteration loop is given in Ref. 5 as

$$\hat{q}_k = [\beta - (1-\beta)(s_k/v_k)]q_k, \qquad k = 1, 2, \ldots, p \tag{27}$$

where $\hat{q}_k$ is the updated kth generalized design variable and q_k is the current kth variable.

We approximately evaluate the updated structural eigenvalue

$$\hat{\lambda}_k = \lambda_k + \frac{\partial \lambda_i}{\partial q}\Delta q \tag{28}$$

where

$$\frac{\partial \lambda_i}{\partial q} = \frac{\partial \lambda_i}{\partial h}C$$

$$\Delta q = (\Delta q_1 \quad \Delta q_2 \quad \ldots \quad \Delta q_p)^t \tag{29}$$

$$= (\hat{q}_1 - q_1, \ldots, \hat{q}_k - q_k, \ldots, \hat{q}_p - q_p)^t$$

Equation (28) is rewritten as

$$(\beta - 1)\frac{\partial \lambda_i}{\partial q}\begin{bmatrix} \ddots & & \\ & \dfrac{s_k}{v_k} & \\ & & \ddots \end{bmatrix}_{k=1\sim p}, \qquad q = \bar{\lambda}_i^{u,l} - \lambda_i - (\beta - 1)\frac{\partial \lambda_i}{\partial q}q \tag{30}$$

Substituting Eqs. (25) and (26) into Eq. (30) obtains a set of linear equations determining the Lagrange multipliers:

$$\sum_{j=1}^{m}\frac{\partial \lambda_i}{\partial q}\begin{bmatrix} \ddots & & \\ & \left(\dfrac{\partial \lambda_j}{\partial h}c_k \Big/ v_k\right) & \\ & & \ddots \end{bmatrix}_{k=1\sim p}$$

$$\times q \cdot (\nu_{jf} - \nu_{jg}) = \frac{1}{\beta - 1}\left(\bar{\lambda}_i^{u,l} - \lambda_i\right) - \frac{\partial \lambda_i}{\partial q}q$$

$$i = 1, 2, \ldots, m \tag{31}$$

IV. Numerical Example

A square flexible plate supported by five springs is given as an example. The geometry and physical parameters are length 3 m, thickness 8 mm, Young's modulus 211 GPa, Poisson's ratio 0.3, and material density 7800 kg/m^3.

The whole plate is divided into nine identical substructures, as shown in Fig. 1. Here, the stiffnesses of the five springs are modifiable. Hence, five substructures, including elastic bearings, are modifiable, i.e., $\beta = 1, 2, \ldots, 5$, whereas the other four are unmodifiable, i.e., $\gamma = 6, \ldots, 9$. Each substructure is divided into 16 plate elements. Thus, the entire structure has 169 nodes, each of which has 3 degrees of freedom; therefore, the entire structure has 507 degrees of freedom. The first three fixed-interface normal modes of each structure are extracted to be synthesized, and the entire structure joint coordinates are 144. Therefore, the condensed structure degrees of freedom is 171. Using modal synthesis techniques, one quickly extracts the required simple or repeated eigenvalue sensitivities and corresponding modes after each iteration in the optimality process.

The objective function is simplified as

$$W(h) = \sum_{i=1}^{5}k_i = \sum_{i=1}^{5}h_i k_{0i} \tag{32}$$

where h_i is the design variable, $k_{01} = k_{02} = k_{03} = k_{04} = 0.3 \times 10^5$ N/m, and $k_{05} = 0.1 \times 10^5$ N/m under frequency constraints (in radian per second)

$$200 \leq \omega_1 \leq 220, \qquad 250 \leq \omega_2 = \omega_3 \leq 300$$

The initial design variable $h_i = 1.0$ $(i = 1, 2, \ldots, 5)$, and the first three frequencies are 167.8, 342.6, and 342.6 rad/s, respectively. When evaluating the Lagrange multipliers, the authors attempted to use the simple frequency method in Ref. 5, but the coefficient matrix exhibited singularity, demonstrating that the method is infeasible.

Again attempting to use the method in Ref. 5, we first artificially give a set of design variable values and let the first three frequencies be simple. The frequency constraints (in radian per second) are adjusted as

$$200 \leq \omega_1 \leq 220, \qquad 230 \leq \omega_2 \leq 250, \qquad 270 \leq \omega_3 \leq 300$$

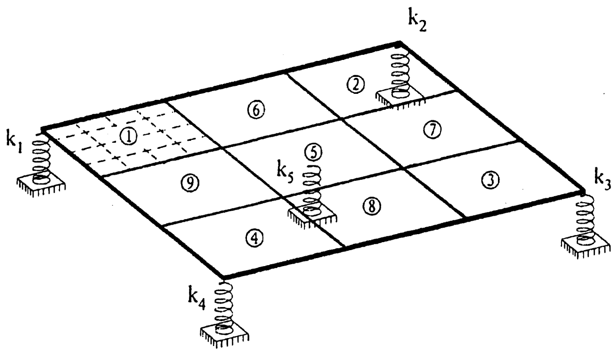

Fig. 1 Square flexible plate supported by five springs with its substructure division.

where

$$a_{ij}^k = z_i^t \left[\frac{\partial \boldsymbol{K}}{\partial h_k} - \theta(h_0) \frac{\partial \boldsymbol{M}}{\partial h_k} \right] z_j, \qquad k = 1, 2, \ldots, n$$

It is the necessary condition that ensures $\delta \boldsymbol{h} \in \Delta \boldsymbol{H}$.

The equivalent matrix expression of Eq. (21a) is

$$\boldsymbol{D}\delta\boldsymbol{h} = \boldsymbol{0} \tag{21b}$$

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Substituting Eq. (22) into the variation of the Lagrange function gives

$$\begin{aligned} \delta L(\boldsymbol{h}, \nu) &= \frac{\partial W}{\partial \boldsymbol{h}} \delta\boldsymbol{h} + \sum_{i=1}^{m} (\nu_{if} - \nu_{ig}) \frac{\partial \lambda_i}{\partial \boldsymbol{h}} \delta\boldsymbol{h} \\ &= \frac{\partial W}{\partial \boldsymbol{h}} \boldsymbol{C} \cdot \delta\boldsymbol{q} + \sum_{i=1}^{m} (\nu_{if} - \nu_{ig}) \frac{\partial \lambda_i}{\partial \boldsymbol{h}} \boldsymbol{C} \cdot \delta\boldsymbol{q} \end{aligned} \tag{23}$$

So far, the Kuhn–Tucker condition can be used:

$$\frac{\partial W}{\partial \boldsymbol{h}} \boldsymbol{C} + \sum_{i=1}^{m} (\nu_{if} - \nu_{ig}) \frac{\partial \lambda_i}{\partial \boldsymbol{h}} \boldsymbol{C} = \boldsymbol{0} \tag{24}$$

with

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$$\begin{aligned} \frac{\partial \lambda_i}{\partial \boldsymbol{q}} &= \frac{\partial \lambda_i}{\partial \boldsymbol{h}} \boldsymbol{C} \\ \Delta\boldsymbol{q} &= (\Delta q_1 \quad \Delta q_2 \quad \ldots \quad \Delta q_p)^t \\ &= (\hat{q}_1 - q_1, \ldots, \hat{q}_k - q_k, \ldots, \hat{q}_p - q_p)^t \end{aligned} \tag{29}$$

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$$(\beta - 1) \frac{\partial \lambda_i}{\partial \boldsymbol{q}} \begin{bmatrix} \ddots & & \\ & \dfrac{s_k}{v_k} & \\ & & \ddots \end{bmatrix}_{k=1\sim p}, \qquad \boldsymbol{q} = \bar{\lambda}_i^{u,l} - \lambda_i - (\beta - 1) \frac{\partial \lambda_i}{\partial \boldsymbol{q}} \boldsymbol{q} \tag{30}$$

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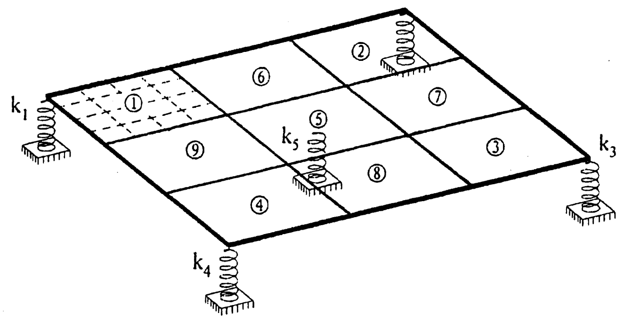

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